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Transient Conjugated Heat Transfer From a Rectangular Hot Film

P. W. Liang* and K. D. Cole†
University of Nebraska-Lincoln, Lincoln, Nebraska 68588

A numerical calculation is carried out for the transient response of a flush-mounted hot-film sensor in a gas flow heated by a step change in the hot-film output flux. The numerical method used is called the unsteady surface element method. This calculation is part of a study of unsteady behavior of hot-film anemometers that involve substrate effects. The numerical results show that the early-time film temperature agrees well with an analytical solution, and the steady-state Nusselt number agrees with an existing numerical solution. The spatial average film temperature vs time is presented for several values of the rectangular hot-film aspect ratio.

Nomenclature

a	=	streamwise half-length of the hot film, m
b	=	spanwise half-length of the hot film, m
K	=	thermal conductivity, W/(mK)
LX_j	=	streamwise length of the jth surface element, m
LY_j	=	spanwise length of the jth surface element, m
M	=	number of time steps
N	=	number of surface elements
N_β	=	flow parameter, Eq. (16)
Nu	=	Nusselt number
P	=	hot-film output flux, W/m^2
Pe	=	Peclet number, $\beta(2a)^2/\alpha_f$
q	=	heat flux, W/m^2
q_0	=	hot-film output, W/m^2
T	=	temperature
T_{av}	=	spatial average hot-film temperature
T_0	=	initial temperature
T^+	=	$(T - T_0)/(q_0 a/K_s)$
t^+	=	$\alpha_s t/a^2$
x	=	streamwise coordinate
x^+	=	x/a
x_j	=	center of the jth surface element
y	=	spanwise coordinate
y^+	=	y/a
y_j	=	center of the jth surface element
z	=	coordinate normal to the interface
α	=	thermal diffusivity, m^2/s
β	=	velocity gradient, s^{-1}
$\Gamma(\ )$	=	gamma function
$\Delta\psi$	=	influence function, Eq. (13)
η	=	dummy space variable in the spanwise direction
λ	=	dummy time variable
$\mu(\ )$	=	unit step function
ξ	=	dummy space variable in the streamwise direction
ψ	=	fundamental solution, defined in Appendix A

Subscripts

av	=	spatial average on hot film
f	=	fluid
k	=	kth time step
s	=	solid
0	=	inital value

Superscripts

f	=	fluid
s	=	solid
+	=	dimensionless variable

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*Graduate Student, Department of Mechanical Engineering.
†Assistant Professor, Department of Mechanical Engineering. Member AIAA.

Introduction

FLUSH-mounted hot-film sensors have been used in unsteady fluid flow for many years. However, there have been few numerical studies of the unsteady heat transfer in these sensors. The numerical calculation reported here is part of a continuing study of unsteady conjugated heat transfer in flush-mounted hot films; that is, hot films for which the substrate participates in the unsteady heat transfer. Substrate participation is important primarily in gas flows.

In the literature most studies of conjugated heat transfer have been limited to steady flow and steady heat transfer. Tanner[1] computed the steady-state mean temperature of the rectangular film in the absence of streamwise and spanwise heat conduction in the flow by a double Fourier-transform method. Kalumuck[2] carried out a three-dimensional calculation by a Fourier-transform method as well to compute the average film temperature vs the thermal conductivity ratio K_s/K_f, the aspect ratio b/a, and the Peclet number of the flow.

For transient problems, many different numerical methods have been employed in solving three-dimensional heat conduction problems with various boundary conditions and material properties.[3–6] In this paper the unsteady surface element (USE) method is applied to the three-dimensional transient conjugated heat transfer problem.

The USE method combines the discretization techniques of the boundary element method with Duhamel's superposition principle. The USE method is well suited for transient conduction problems in dissimilar bodies that are thermally connected to one another because only the interface between the connected bodies is divided into elements.[7]

Cole and Beck[8,9] applied the USE method to analyze a two-dimensional strip heater on the fluid/solid interface. In one study,[8] the fluid flow was steady and the heat transfer was unsteady. In the second study,[9] both the flow and the heat transfer were unsteady. The output heat flux from the strip heater was chosen to maintain a constant spatial-average temperature to simulate a constant-temperature anemometer.

In the present paper the USE method is extended to three spatial dimensions. To the author's knowledge there have been no fully three-dimensional applications of the USE method. The geometry of interest is a rectangular hot film on the fluid/solid interface. The fluid flow is steady but the heat transfer is unsteady. Refer to Fig. 1. The three-dimensional calculation is motivated by a commercially available quartz-substrate hot-film sensor with an aspect ratio b/a of about 7. Other important applications are: heat removal from electronic circuitry, surface hardening and heat-treating proc-

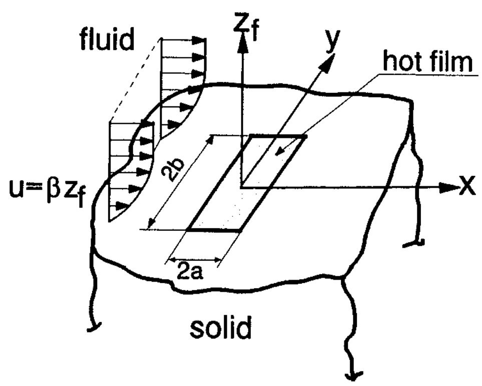

Fig. 1 Geometry of a rectangular hot film on the fluid/solid interface.

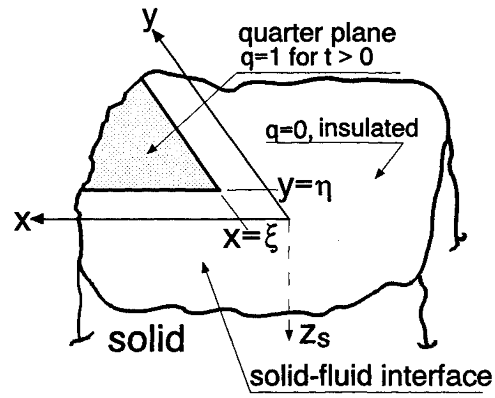

Fig. 2 Semi-infinite geometry for solid fundamental solution.

esses, and problems for which the interface conditions are not known.

In this paper the mathematical model and the USE method formulation are presented for the three-dimensional conjugated heat transfer problem. The numerical results are compared with both an analytical solution of the early-time response and the steady-state solution which is available in the literature. The response of the hot-film sensor over a wide range of time is also presented for several values of the hot-film aspect ratio. The numerical solution will apply to the analysis of commercially available hot-film anemometers.

Mathematical Model

Figure 1 shows the geometry of the conjugated heat transfer problem—a fluid flows over the surface of a solid body in which a thin rectangular hot film is embedded. The energy equation of this problem is split up into two domains: solid and fluid. The thermal properties of the problem are constant and the buoyant flow effect is neglected in deriving the energy equations. In addition, the hot film is so thin that the thermal storage in the hot film may be neglected.

Energy Equation and Temperature Expression in the Solid

The three-dimensional heat conduction problem in the solid is described by the following equations:

$$\alpha_s\left(\frac{\partial^2 T^s}{\partial x^2}+\frac{\partial^2 T^s}{\partial y^2}+\frac{\partial^2 T^s}{\partial z_s^2}\right)=\frac{\partial T^s}{\partial t}\qquad \begin{matrix}-\infty<x<\infty\\ -\infty<y<\infty\\ 0\le z_s<\infty\end{matrix}\tag{1}$$

$$T^s(x, y, z_s, t=0)=T_0\qquad \text{for } t=0\tag{2}$$

$$-K_s\frac{\partial T^s}{\partial z_s}(x, y, z_s=0, t)=q^s(x, y, t)$$

$$\text{on boundary } z_s=0\tag{3}$$

where $T^s(x, y, z_s, t)$ is the unknown temperature in the solid, and $q^s(x, y, t)$ is the unknown heat flux at the interface in the $+z_s$ direction.

The temperature expression $T^s(x, y, z_s, t)$ can be found by Duhamel's superposition principle. Let $\psi^s(x-\xi, y-\eta, z_s, t)$ be the temperature rise in the solid due to a unit step input of the heat flux over the quarter-plane $x>\xi$, $y>\eta$, and insulated elsewhere. Refer to Fig. 2. The temperature in the solid is given by

$$T^s(x, y, z_s, t)-T_0=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\lambda=0}^{t}q^s(\xi, \eta, \lambda)\frac{\partial^3\psi^s}{\partial t\partial\xi\partial\eta}\times\partial\xi\partial\eta\partial\lambda\tag{4}$$

A derivation of Eq. (4) is given in Appendix A.

Energy Equation and Temperature Expression in the Fluid

The energy equation in the fluid describes the forced convection of a constant-property fluid flowing over a flat plate. It is given by the following equation:

$$\beta z_f\frac{\partial T^f}{\partial x}=\alpha_f\frac{\partial^2 T^f}{\partial z_f^2}\qquad \begin{matrix}-\infty<x<\infty\\ -\infty<y<\infty\\ 0\le z_f<\infty\end{matrix}\tag{5}$$

$$T^f(x, y, z_f, t=0)=T_0\qquad \text{for } t=0\tag{6}$$

$$-K_f\frac{\partial T^f}{\partial z_f}(x, y, z_f=0, t)=q^f(x, y, t)$$

$$\text{on boundary } z_f=0\tag{7}$$

where $T^f(x, y, z_f, t)$ is the unknown temperature in the fluid, and $q^f(x, y, t)$ is the unknown heat flux at the interface in the $+z_f$ direction.

This simple equation is useful under the following conditions: the fluid is a gas and the substrate is quartz ($K_s >> K_f$); no fluid flows in the spanwise direction; the effects of streamwise and spanwise heat conduction are neglected; and the temperature rise on the hot film is small such that the thermal boundary layer lies within the linear portion of the velocity boundary layer.

The temperature expression $T^f(x, y, z_f, t)$ can be found by Duhamel's superposition principle. Let $\psi^f(x-\xi, y-\eta, z_f, t)$ be the temperature rise in the fluid due to a unit step input of heat flux over the quarter-plane $x>\xi$, $y>\eta$, and insulated elsewhere. The temperature in the fluid is given by

$$T^f(x, y, z_f, t)-T_0=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\lambda=0}^{t}q^f(\xi, \eta, \lambda)\frac{\partial^3\psi^f}{\partial t\partial\xi\partial\eta}\times\partial\xi\partial\eta\partial\lambda\tag{8}$$

Derivation of Eq. (8) is given in Appendix A.

Matching Conditions

The temperature and the heat flux matching conditions at the interface are given by

$$T^s(x, y, z_s=0, t)=T^f(x, y, z_f=0, t)\tag{9}$$

$$P(x, y, t)=q^s(x, y, t)+q^f(x, y, t)\qquad \text{for } z_s=z_f=0\tag{10}$$

where $P(x, y, t)$ is the heat flux introduced by the hot film. Heat flux $P(x, y, t)$ drives the heat transfer problem, and some of the introduced heat flows into the fluid (q^f) and some flows into the solid (q^s). Heat flux $P(x, y, t)$ is zero on the region outside the hot film.

The matching conditions are based on two assumptions: 1) the thickness of the hot film is small and the thermal conductivity is high (no thermal storage and no temperature change in the direction normal to the interface); and 2) the hot film, the fluid, and the solid are in perfect thermal contact.

Unsteady Surface Element Formulation

In order to formulate the problem with the surface element method, Eqs. (4) and (8–10) must be approximated in two ways. First, the infinite interface over which heat is transferred between the solid and the fluid must be truncated. [This is equivalent to truncating the infinite limits on the integrals on Eq. (4) and Eq. (8).] We define the active interface as the truncated portion of the infinite interface. If the active interface is sufficiently large, then the truncation error (or the "tail" of the integral) will be small enough to be neglected. Second, the active interface must be discretized into a finite number of surface elements.

Surface Elements

In the present study the active interface is divided into N rectangular surface elements. The heat flux is assumed to be constant on each surface element. The temperature is calculated at the node located on the center of each element j.

The solid temperature given by Eq. (4) needs to be rewritten in a discretized form, that is, in terms of a summation over the surface elements. If (x_j, y_j) is the center of element j with length LX_j and width LY_j, then the limits on the spatial integrals for each element are $(x_j - LX_j/2)$, $(x_j + LX_j/2)$, $(y_j - LY_j/2)$, and $(y_j + LY_j/2)$. The heat flux which is constant over each surface element can be taken outside each spatial integral so Eq. (4) can be written

$$T^s(x, y, z_s, t) - T_0 = \int_{\lambda=0}^{t} \partial\lambda \sum_{j=1}^{N} \times \left[q^s(x_j, y_j, \lambda) \int_{xj-(LX_j/2)}^{xj+(LX_j/2)} \int_{yj-(LY_j/2)}^{yj+(LY_j/2)} \times \frac{\partial^3 \psi^s}{\partial\xi\partial\eta\partial t}(x - \xi, y - \eta, z_s, t - \lambda)\partial\xi\partial\eta \right] \tag{11}$$

The spatial integrals over each surface element can be evaluated to give

$$T^s(x, y, z_s, t) - T_0 = \sum_{j=1}^{N} \int_{\lambda=0}^{t} q^s(x_j, y_j, \lambda) \times \frac{\partial}{\partial t}[\Delta\psi^s(x, x_j, y, y_j, z_s, t - \lambda)]\partial\lambda \tag{12}$$

where $\Delta\psi^s$ is called solid influence function and is defined by

$$\begin{aligned}\Delta\psi^s(x, x_j, y, y_j, z_s, t) &= \psi^s\left[x - \left(x_j + \frac{LX_j}{2}\right), y - \left(y_j + \frac{LY_j}{2}\right), z_s, t\right] \\ &- \psi^s\left[x - \left(x_j - \frac{LX_j}{2}\right), y - \left(y_j + \frac{LY_j}{2}\right), z_s, t\right] \\ &- \psi^s\left[x - \left(x_j + \frac{LX_j}{2}\right), y - \left(j_j - \frac{LY_j}{2}\right), z_s, t\right] \\ &+ \psi^s\left[x - \left(x_j - \frac{LX_j}{2}\right), y - \left(y_j - \frac{LY_j}{2}\right), z_s, t\right]\end{aligned} \tag{13}$$

Four quarter-plane source solutions are superposed into a rectangular source solution as shown in Eq. (13), therefore, $\Delta\psi^s(x, x_j, y, y_j, z, t)$ represents the temperature distribution in the semi-infinite solid due to a unit input of boundary heat flux on a rectangular region $(x_j - LX_j/2) < x < (x_j + LX_j/2)$ and $(y_j - LY_j/2) < y < (y_j + LY_j/2)$, and insulated elsewhere. The solution of this solid influence function is obtainable from a Green's function solution.[10] Only the temperature on the interface ($z_s = 0$) is needed; it is found to be

$$\begin{aligned}\Delta\psi^s(x, x_j, y, y_j, z_s = 0, t) &= \frac{\alpha_s}{2\sqrt{\pi}K_s} \int_{\lambda=0}^{t} \frac{1}{\sqrt{4\alpha_s(t-\lambda)}} \left[\operatorname{erf}\left(\frac{x - x_j + \frac{LX_j}{2}}{\sqrt{4\alpha_s(t-\lambda)}}\right) - \operatorname{erf}\left(\frac{x - x_j - \frac{LX_j}{2}}{\sqrt{4\alpha_s(t-\lambda)}}\right)\right] \\ &\times \left[\operatorname{erf}\left(\frac{y - y_j + \frac{LY_j}{2}}{\sqrt{4\alpha_s(t-\lambda)}}\right) - \operatorname{erf}\left(\frac{y - y_j - \frac{LY_j}{2}}{\sqrt{4\alpha_s(t-\lambda)}}\right)\right]\partial\lambda\end{aligned} \tag{14}$$

A numerical integration method is used to evaluate the time integral in Eq. (14). The singularity at the $\lambda = t$ point is eliminated by a simple change of variable.

The fluid temperature, Eq. (8), is discretized in a fashion similar to the solid temperature. Equation (8) is discretized to give

$$T^f(x, y, z_f, t) - T_0 = \sum_{j=1}^{N} \int_{\lambda=0}^{t} q^f(x_j, y_j, \lambda) \times \frac{\partial}{\partial t}[\Delta\psi^f(x, x_j, y, y_j, z_f, t - \lambda)]\partial\lambda \tag{15}$$

where $\Delta\psi^f$ is defined in a fashion analogous to Eq. (13).

The solution of $\Delta\psi^f$ is obtainable from the two-dimensional influence function given by Bird et al.[11] if extending the line element to plane element. It is expressed as

$$\Delta\psi^f(x, x_j, y, y_j, z_f = 0, t) = \frac{9^{1/3}}{\Gamma\left(\frac{2}{3}\right)} \frac{a}{K_s N_\beta} \times \frac{\left(x - x_j + \frac{LX_j}{2}\right)^{1/3} - \left(x - x_j - \frac{LX_j}{2}\right)^{1/3}}{a^{1/3}} \times \mu(t)\mu(x - x_j)\left(\frac{Y_{j\to i}}{LY_i}\right) \tag{16}$$

where

$$N_\beta = \frac{K_f}{K_s}\left(\frac{\beta a^2}{a_f}\right)^{1/3}$$

$Y_{j\to i}$ is the projection of element j on the element i in the streamwise direction.

Equation (16) yields the temperature rise in the fluid at the location (x, y) on the interface due to a unit step heat flux input on the rectangular surface element j, and insulated elsewhere.

USE Equation

The USE equation is formed by combining the fluid and the solid integral equations with the matching conditions. That is, for a specific point (x_i, y_i) at the center of a surface element, the USE equation is obtained by substituting Eq. (10), (12), and (15) into the temperature matching condition, Eq. (9).

$$\sum_{j=1}^{N} \int_0^t q^f(x_j, y_j, \lambda) \frac{\partial}{\partial t} [\Delta\psi^f(x_i, x_j, y_i, y_j, 0, t-\lambda)]\partial\lambda$$

$$= \sum_{j=1}^{N} \int_0^t [P(x_j, y_j, \lambda) - q^f(x_j, y_j, \lambda)]$$

$$\times \frac{\partial}{\partial t} [\Delta\psi^s(x_i, x_j, y_i, y_j, 0, t-\lambda)]\partial\lambda \qquad (17)$$

This is the USE equation for $i = 1, 2, \ldots, N$. The histories of heat flux, surface temperature, and heat input on the interface are represented in discrete forms by $q(x_j, y_j, t)$, $T(x_j, y_j, z = 0, t)$, and $P(x_j, y_j, t)$, respectively, for $j = 1, 2, \ldots, N$. This set of N equations can be solved for N heat flux histories $q^f(x_j, y_j, t)$ because $\Delta\psi$ and P are known functions. After the heat flux histories are found the temperature values at each surface element may be evaluated by Eq. (12) or Eq. (15).

Numerical Solution

In order to express the USE equation in a matrix form and to solve it by a computer, each time integral in Eq. (17) is treated as a sum of integrals, one integral for each time step. The heat flux is piecewise constant during each time step. The complete derivation of a matrix form of the USE equation is given by Cole and Beck.[8] Here, the matrix form of the USE equation is simply stated as follows:

$$(\bar{\bar{\Phi}}_1^s + \bar{\bar{\Phi}}_1^f)\bar{q}_M^f = \sum_{k=1}^{M-1} (\bar{\bar{\Phi}}_{M-k+1}^s - \bar{\bar{\Phi}}_{M-k}^s)(\bar{P}_k - \bar{q}_k^f)$$

$$- \sum_{k=1}^{M-1} (\bar{\bar{\Phi}}_{M-k+1}^f - \bar{\bar{\Phi}}_{M-k}^f)\bar{q}_k^f + \bar{\bar{\Phi}}_1^s \bar{P}_M \qquad (18)$$

where $\bar{\bar{\Phi}}_k^s$, $\bar{\bar{\Phi}}_k^f$ are the matrices whose components are

$$(\bar{\bar{\Phi}}_k^s)_{ij} = -\Delta\psi^s(x_i, x_j, y_i, y_j, z_s = 0, t_k)$$

$$(\bar{\bar{\Phi}}_k^f)_{ij} = -\Delta\psi^f(x_i, x_j, y_i, y_j, z_f = 0, t_k) \qquad (19)$$

$\bar{q}_k^f$, $\bar{P}_k$ are column vectors whose components are

$$(\bar{q}_k^f)_j = q^f(x_j, y_j, t_k)$$
$$(\bar{P}_k)_j = P(x_j, y_j, t_k) \qquad (20)$$

Results and Discussion

The early-time response of a commercially available hot-film sensor (TSI sensor #1237) subjected to a unit step heat input on the hot film was computed by the numerical USE method. The results for spatial-average dimensionless temperatures on the hot film are compared to a one-node analytical solution in Table 1. The results shown in Table 1 are for the following: a hot film with $a = 7 \times 10^{-5}$ m and $b/a = 7.14$; air flow with $\beta = 48570$ s^{-1}; and, a quartz substrate with $K_s = 1.425$ W/mK and $\alpha_s = 8.87 \times 10^{-7}$ m^2/s. The numerical solution used 96 surface elements placed inside the hot film with smaller elements near the edge of the hot film. The analytical solution which is discussed in Appendix B is very useful for the early-time response (the time before the interface adjacent to the hot film has heat transfer effects).

Table 1 Early-time results for spatial average temperature on the hot film (quartz-substrate hot film) ($N_\beta = 0.0405$, $K_s/K_f = 54.5$, $b/a = 7.14$, $\alpha_f/\alpha_s = 24.92$)

t^+	Analytical Solution	Numerical Solution	Percent Difference
0.005	0.07785	0.07825	0.52
0.010	0.10896	0.10958	0.57
0.020	0.15185	0.15284	0.65
0.030	0.18388	0.18512	0.65
0.060	0.25342	0.25511	0.67
0.090	0.30420	0.30622	0.67
0.120	0.34530	0.34762	0.67
0.150	0.38023	0.38283	0.68
0.180	0.41079	0.41364	0.70
0.210	0.43803	0.44113	0.71
0.250	0.47039	0.47378	0.72

Table 2 Steady-state Nusselt number as a function of Pe, b/a, and K_s/K_f

Pe	b/a	K_s/K_f	Kalumuck (1983)	USE method	Percent difference
16	1.0	50.0	108.00	108.55	0.50
256	1.0	50.0	112.30	112.72	0.37
16	1.0	20.0	45.55	45.07	1.06
64	1.0	20.0	46.91	46.64	0.58
16	1.0	5.0	14.00	13.22	5.57
64	1.0	5.0	15.24	14.63	3.97
256	1.0	5.0	17.33	16.82	2.93
16	5.0	11.0	14.83	14.71	0.82
64	5.0	11.0	16.27	16.14	0.77
144	5.0	11.0	17.43	17.32	0.61
16	8.0	4.1	6.57	6.44	2.01
64	8.0	4.1	7.82	7.80	0.25
16	∞	5.0	5.94	5.82	2.02
64	∞	5.0	7.44	7.33	1.48
144	∞	5.0	8.63	8.53	1.16

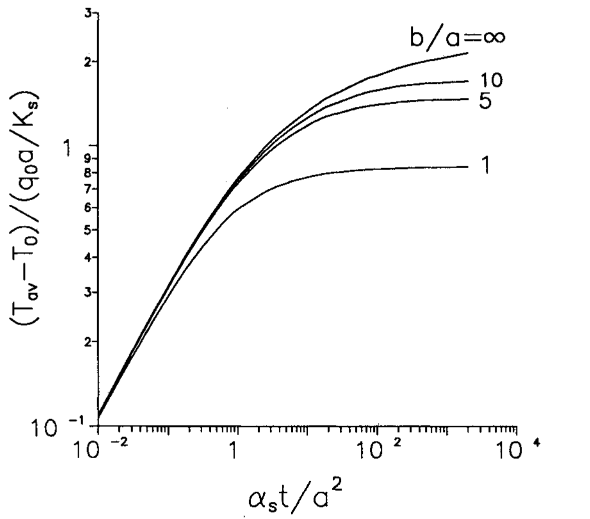

Fig. 3 Spatial average temperature on the hot film vs time for various hot-film aspect ratio, and for $N_\beta = 0.156$.

The two solutions agree within 0.72% for the dimensionless time less than 0.25 (0.0024 s). This provides a check on the computation procedure of the USE method.

Figure 3 shows numerical results for dimensionless temperature vs time for a wide range of dimensionless times and for aspect ratio $b/a = 1$, 5, 10, and ∞. These results were calculated with surface elements placed both inside and outside the hot film.

The spatial average temperature on the hot film depends on the three dimensionless parameters: t^+, b/a, and N_β; that is, $T_{av}^+ = T_{av}^+(t^+, b/a, N_\beta)$. Figure 3 shows T_{av}^+ vs t^+ for various aspect ratios b/a and for $N_\beta = 0.156$. When t^+ is small, the curves for the larger b/a values tend to lie on the curve $T^+_{av} \sim (t)^{1/2}$, which is the curve for one-dimensional heat conduction into the solid. It is only after t^+ is large ($t^+ > 1$) that convection effects and multidimension conduction effects come into play. Figure 3 also shows that reducing the b/a value lowers the steady-state temperature and shortens the time needed to reach steady state. The case $b/a = \infty$ has been approximated by a very large aspect ratio, $b/a = 500$.

To form each curve on Fig. 3, several computer runs with different time steps were used, with larger time step sizes for the larger time temperature responses. A typical curve on Fig. 3 required about 800 CPU s on the VAX 8800 computer at the University of Nebraska.

Without loss of generality, the steady-state Nusselt number is defined as

$$Nu = \frac{q_0(2a)}{(T_{av} - T_0)K_f} = \frac{2K_s}{T^+K_f} \tag{21}$$

For various values of Pe, b/a, and K_s/K_f, the steady-state Nusselt numbers are presented in Table 2 together with the values calculated by Kalumuck.[2] The steady-state results were computed with the USE method by using several very large time steps. Kalumuck used a spatial Fourier transform method with a numerical transform inversion. In Table 2, the results show that the Nusselt number is increased by increasing the K_s/K_f or Pe and the percent difference between the two methods decreases as the Pe increases and/or as K_s/K_f increases. In the general, the difference is less than 2.02% except for the case $K_s/K_f = 5$ and $b/a = 1$. The two methods have larger differences in the case $K_s/K_f = 5$ and $b/a = 1$, since Kalumuck includes both the streamwise and the spanwise conduction heat in the fluid but the present calculation does not. The effect of spanwise conduction of heat in the fluid is small whenever the aspect ratio is large; for example, the difference is lowered to less than 2.02% for the case $K_s/K_f = 5$ and $b/a = \infty$. These results also support the validity of neglecting the fluid spanwise and streamwise heat conduction when $K_s/K_f = 54$ in the air/quartz case.

Based on studies of the effects of the discretization parameters which include time-step sizes, active interface lengths, and element sizes, the accuracy is about 0.7% for $b/a \geq 1$ and $K_s/K_f \geq 54$.

Conclusions

The USE method has been extended to a fully three-dimensional geometry for transient conjugated heat transfer from a rectangular hot film. The early-time response of a commercially available hot-film sensor computed by the numerical USE method is in good agreement with the analytical USE method for a unit step heat input on the hot film. The spatial average temperatures on the hot film are presented for a wide range of time for various values of the hot-film aspect ratio. For larger values of the hot-film aspect ratio, the three-dimensional results are consistent with the two-dimensional solution. A parametric study of the steady-state results for several different values of hot-film aspect ratio, conductivity ratio, and Peclet number agree well with previous numerical solutions.

The three-dimensional approach is capable of computing the transient responses of a rectangular hot-film anemometer in the air flow. This approach reproduces the cause and effect of actual sensors: heat is introduced at the hot film, and the temperature rises as a result of the introduced heat. The reported results are for a simple step change in the hot-film heat flux; however, any time-varying heat flux could have been used. The present work is a first step toward the analysis of unsteady air flow over a constant-temperature hot film, which is important for the shear stress sensor problem.

Appendix A: Derivation of Eqs. (4) and (8)

The fundamental solution $\psi(x - \xi, y - \eta, z, t)$ in Eqs. (4) and (8) is the resulting temperature distribution due to a unit step input of heat flux over the quarter-plane $x > \xi$, $y > \eta$ at $z = 0$, and insulated elsewhere for $t > 0$. Here, a dummy variable λ is introduced and t is replaced by $t - \lambda$, therefore $\psi(x - \xi, y - \eta, z, t - \lambda)$ is the new variable to represent the resulting temperature for $t > \lambda$ and is assumed to be zero for $t < \lambda$. Next, suppose that the heat flux is raised abruptly to a value $q(x_j, y_j, t = 0)$ on the small area centered at location (x_j, y_j) and hold this value until $t = \lambda$, then again the heat flux is raised abruptly to a new value $q(x_j, y_j, t = \lambda_1)$ by an amount $q(x_j, y_j, t = \lambda_1) - q(x_j, y_j, t = 0)$ and so on. Therefore, the temperature distribution at $t = \lambda_n$ can be found approximately by superposing the temperature raised due to n number steps of heat flux input over each small area $\Delta\eta\Delta\xi$ on the $z = 0$ boundary.

$$\begin{aligned} T(x, y, z, t) - T_0 = \sum_{j=1}^{\infty} \lim_{\Delta\xi\to 0} \lim_{\Delta\eta\to 0} \Bigg[& q_j(t = 0) \frac{\Delta^2\psi_j(x - \xi, y - \eta, t)}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta + [q_j(t = \lambda_1) \\ & - q_j(t = 0)] \frac{\Delta^2\psi_j(x - \xi, y - \eta, z, t - \lambda_1)}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta \\ & + [q_j(t = \lambda_2) - q_j(t = \lambda_1)] \\ & \times \frac{\Delta^2\psi_j(x - \xi, y - \eta, z, t - \lambda_2)}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta \\ & + \dots\dots\dots + \quad \dots\dots\dots + + [q_j(t = \lambda_n) \\ & - q_j(t = \lambda_{n-1})] \frac{\Delta^2\psi_j(x - \xi, y - \eta, z, t - \lambda_n)}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta \Bigg] \end{aligned} \tag{A1}$$

where

$$\begin{aligned} q_j(t) &= q(x_j, y_j, t)\, \Delta^2\psi_j(x - \xi, y - \eta, z, t - \lambda) \\ &= \psi\left[x - \left(x_j + \frac{\Delta\xi}{2}\right), y - \left(y_j + \frac{\Delta\eta}{2}\right), z, t - \lambda\right] \\ &- \psi\left[x - \left(x_j - \frac{\Delta\xi}{2}\right), y - \left(y_j + \frac{\Delta\eta}{2}\right), z, t - \lambda\right] \\ &- \psi\left[x - \left(x_j + \frac{\Delta\xi}{2}\right), y - \left(y_j - \frac{\Delta\eta}{2}\right), z, t - \lambda\right] \\ &+ \psi\left[x - \left(x_j - \frac{\Delta\xi}{2}\right), y - \left(y_j - \frac{\Delta\eta}{2}\right), z, t - \lambda\right] \end{aligned}$$

Letting $(\Delta q_j)_k = q_j(\lambda_{k+1}) - q_j(\lambda_k)$ and $\Delta\lambda_k = \lambda_{k+1} - \lambda_k$. Eq. (A1) can be written as

$$\begin{aligned} T(x, y, z, t) - T_0 = \lim_{\Delta\xi\to 0} \lim_{\Delta\eta\to 0} \sum_{j=1}^{\infty} \Bigg[& q_j(t = 0) \frac{\Delta^2\psi_j(x - \xi, y - \eta, z, t)}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta \\ & + \sum_{k=0}^{n-1} \left(\frac{(\Delta q_j)_k}{\Delta\lambda_k}\right) \Delta\lambda_k \\ & \times \frac{\Delta^2\psi_j[x - \xi, y - \eta, z, t - \lambda_{k+1}]}{\Delta\xi\Delta\eta} \Delta\xi\Delta\eta \Bigg] \end{aligned} \tag{A2}$$

Taking $\lim(\Delta\lambda_k \to 0)$, Eq. (A2) can be written in the integral form as

$$T(x, y, z, t) - T_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[q(\xi, \eta, t = 0) \times \frac{\partial^2\psi(x - \xi, y - \eta, z, t)}{\partial\xi\partial\eta} + \int_{\lambda=0}^{t}\left(\frac{\partial q(\xi, \eta, \lambda)}{\partial\lambda}\right) \times \partial\lambda \frac{\partial^2\psi(x - \xi, y - \eta, z, t - \lambda)}{\partial\xi\partial\eta}\right]\partial\xi\partial\eta \tag{A3}$$

Integrating the time integral in the above equation by parts and applying the initial condition yield

$$T(x, y, z, t) - T_0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\lambda=0}^{t} q(\xi, \eta, \lambda) \times \frac{\partial^3\psi(x - \xi, y - \eta, z, t - \lambda)}{\partial t\partial\xi\partial\eta}\partial\lambda\partial\xi\partial\eta \tag{A4}$$

Equation (A4) is the same as Eqs. (4) and (8).

Appendix B: One-Node Analytical Solution

The set of simultaneous integral equations represented by Eq. (17) can be solved analytically for small times in the case of a single surface element that corresponds to the hot-film location. The following 3-D analysis is parallel to the 2-D analysis of Cole and Beck.[8]

Consider the spatial average temperature on the hot film, given by

$$T_{\text{av}}^{+}(t^+) = \int_0^{t^+} q^f(\lambda^+)\frac{\partial}{\partial t^+}[\Delta\psi^f(t^+ - \lambda^+)]\partial\lambda^+ \tag{B1}$$

where the influence function $\Delta\psi^f$ is now the average temperature on the hot film caused by a unit step in heat flux. For a single surface element, the USE equation reduces to a single integral equation

$$\int_0^{t^+} q^f(\lambda^+)\frac{\partial}{\partial t^+}[\Delta\psi^f(t^+ - \lambda^+)]\partial\lambda^+ = \int_0^{t^+}[P(\lambda^+) - q^f(\lambda^+)]\frac{\partial}{\partial t^+}[\Delta\psi^s(t^+ - \lambda^+)]\partial\lambda^+ \tag{B2}$$

where the x dependence and y dependence have been absorbed into the spatial average influence functions $\Delta\psi^f$ and ...ons (B1) and (B2) can be solved with the Laplace ...method to give the transformed spatial average ...ure in the form[8]

$$T_{\text{av}}^{*}(\theta) = \frac{\theta P^*(\theta)[\Delta\psi^s(\theta)]^*[\Delta\psi^f(\theta)]^*}{[\Delta\psi^f(\theta)]^* + [\Delta\psi^s(\theta)]^*} \tag{B3}$$

where θ is the Laplace transform parameter and $(\cdot)^*$ is the Laplace tranform. The spatial average influence function $\Delta\psi^f$ is given by Cole and Beck,[8] and the Laplace transform of $\Delta\psi^f$ is

$$[\Delta\psi^f(\theta)]^* = \frac{a}{K_s C N_\beta \theta} \tag{B4}$$

The spatial average influence function for the rectangular hot-film region is given by[12]

$$\Delta\psi^s(t) = \frac{a}{K_s}\left[2\left(\frac{t^+}{\pi}\right)^{1/2} - \left(1 + \frac{1}{b^+}\right)\left(\frac{t^+}{\pi}\right) + \frac{2}{3b^+}\left(\frac{t^+}{\pi}\right)^{3/2}\right] \tag{B5}$$

and its Laplace transform is

$$[\Delta\psi^s(\theta)]^* = \frac{a}{\theta K_s \pi}\left[\pi\theta^{-1/2} - \left(1 + \frac{1}{b^+}\right)\theta^{-1} + \frac{1}{(2b^+)}\theta^{-3/2}\right] \tag{B6}$$

where $b^+ = b/a$, the aspect ratio of the hot film.

A step input for the hot film is given by $P(t) = q_0\mu(t)$, where $\mu(t)$ is the unit step function, and the Laplace transform of this step input is $P^*(\theta) = q_0/\theta$. Replace these expressions for $[\Delta\psi^f(\theta)]^*$, $[\Delta\psi^s(\theta)]^*$, and $P^*(\theta)$ into Eq. (B3) to get

$$T_{\text{av}}^{*}(\theta) = \frac{q_0 a \varepsilon}{\theta K_s C N_\beta}\left(\frac{1}{1 + \varepsilon}\right) \tag{B7}$$

where

$$\varepsilon = CN_\beta\left[\theta^{-1/2} - \left(1 + \frac{1}{b^+}\right)\frac{1}{\pi}\theta^{-1} + \frac{1}{2b^+\pi}\theta^{-3/2}\right] \tag{B8}$$

To inverse transform this expression for small times, we use the fact that $1/\theta << 1$ when t^+ is small. Because $1/\theta$ is small and because $CN_\beta < 1$ for the air/quartz case, then ε is also small. The ratio $1/(1 + \varepsilon)$ can be replaced by the binomial expansion to give

$$T_{\text{av}}^{*}(\theta) = \frac{q_0 a \varepsilon(1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \ldots)}{\theta K_s C N_\beta} \tag{B9}$$

When multiplied out the resulting expression can be inverse transformed term by term. The final result for retaining terms up to $1/\theta^3$ is

$$\frac{T_{\text{av}}(t) - T_0}{(q_0 a/K_s)} = 2(t^+/\pi)^{1/2} - t^+(A + CN_\beta) + \frac{4}{3}\frac{1}{\sqrt{\pi}}(t^+)^{3/2}[B + 2ACN_\beta + (CN_\beta)^2] - \frac{1}{2}(t^+)^2[2BCN_\beta - A^2CN_\beta + 3A(CN_\beta)^2 + (CN_\beta)^3] \tag{B10}$$

where $A = (1 + 1/b^+)/\pi$, $B = 1/(2b^+\pi)$, $C = (2/3)^{5/3}\Gamma(2/3) = 0.6889$, and $N_\beta = K_f/K_s\,(\beta a^2/\alpha_f)^{1/3}$.

This approximate expression is limited to small time, $\alpha_s t/a^2 < 0.3$. In the limit as $b^+ \to \infty$ this expression reduces to the 2-D case. In the limit as $N_\beta \to 0$ this expression reduces to the average temperature on the heated region of the solid only (no flow) given by Keltner et al.[10]

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